The Radicals of Hopf Module Algebras *

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Abstract

The characterization of H-prime radical is given in many ways. Meantime, the relations between the radical of smash product $R\#H$ and the H-radical of Hopf module algebra R are obtained.

0 Introduction and Preliminaries

In this paper, let k be a commutative associative ring with unit, H be an algebra with unit and comultiplication $\triangle$ (i.e. Δ is a linear map: $H \to H \otimes H$), R be an algebra over k (R may be without unit) and R be an H-module algebra.

We define some necessary concept as follows.

If there exists a linear map $\begin{cases} H \otimes R & \longrightarrow R \\ h \otimes r & \mapsto h \cdot r \end{cases}$ such that

$$h \cdot rs = \sum (h_1 \cdot r)(h_2 \cdot s) \quad \text{and} \quad 1_H \cdot r = r$$

for all $r, s \in R, h \in H$, then we say that H weakly acts on R. For any ideal I of R, set

$$(I : H) := \{x \in R \mid h \cdot x \in I \text{ for all } h \in H\}.$$

I is called an H-ideal if $h \cdot I \subseteq I$ for any $h \in H$. Let I_H denote the maximal H-ideal of R in I. It is clear that $I_H = (I : H)$. An H-module algebra R is called an H-simple module algebra if R has not any non-trivial H-ideals and $R^2 \neq 0$. R is said to be H-semiprime if there are no non-zero nilpotent H-ideals in R. R is said to be H-prime if $IJ = 0$ implies $I = 0$ or $J = 0$ for any H-ideals I and J of R. An H-ideal I is called an H-(semi)prime ideal of R if R/I is H-(semi)prime. $\{a_n\}$ is called an H-m-sequence in R with beginning a if there exist $h_n, h'_n \in H$ such that $a_1 = a \in R$ and $a_{n+1} = (h_n.a_n)b_n(h'_n.a_n)$ for any

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natural number n. If every H-m-sequence $\{a_n\}$ with $a_{7.1.1} = a$, there exists a natural number k such that $a_k = 0$, then a is called an H-m-nilpotent element. Set

$$W_H(R) = \{a \in R \mid a \text{ is an } H\text{-}m\text{-nilpotent element}\}.$$

R is called an H-module algebra if the following conditions hold:

(i) R is a unital left H-module(i.e. R is a left H-module and $1_H \cdot a = a$ for any $a \in R$);

(ii) $h \cdot ab = \sum(h_1 \cdot a)(h_2 \cdot b)$ for any $a, b \in R$, $h \in H$, where $\Delta(h) = \sum h_1 \otimes h_2$.

H-module algebra is sometimes called a Hopf module.

If R is an H-module algebra with a unit 1_R, then

$$h \cdot 1_R = \sum_h (h_1 \cdot 1_R)(h_2 S(h_3) \cdot 1_R)$$

$$= \sum_h h_1 \cdot (1_R(S(h_2) \cdot 1_R)) = \sum_h h_1 S(h_2) \cdot 1_R = \epsilon(h) 1_R,$$

i.e. $\quad h \cdot 1_R = \epsilon(h) 1_R$

for any $h \in H$.

An H-module algebra R is called a unital H-module algebra if R has a unit 1_R such that $h \cdot 1_R = \epsilon(h)1_R$ for any $h \in H$. Therefore, every H-module algebra with unit is a unital H-module algebra. A left R-module M is called an R-H-module if M is also a left unital H-module with $h(am) = \sum(h_1 \cdot a)(h_2 m)$ for all $h \in H, a \in R, m \in M$. An R-H-module M is called an R-H- irreducible module if there are no non-trivial R-H-submodules in M and $RM \neq 0$. An algebra homomorphism $\psi : R \to R'$ is called an H-homomorphism if $\psi(h \cdot a) = h \cdot \psi(a)$ for any $h \in H, a \in R$. Let r_b, r_j, r_l, r_{bm} denote the Baer radical, the Jacobson radical, the locally nilpotent radical, the Brown-MacCoy radical of algebras respectively. Let $I \lhd_H R$ denote that I is an H-ideal of R.

1 The H-special radicals for H-module algebras

J.R. Fisher [7] built up the general theory of H-radicals for H-module algebras. We can easily give the definitions of the H-upper radical and the H-lower radical for H-module algebras as in [11]. In this section, we obtain some properties of H-special radicals for H-module algebras.

Lemma 1.1 *(1) If R is an H-module algebra and E is a non-empty subset of R, then $(E) = H \cdot E + R(H \cdot E) + (H \cdot E)R + R(H \cdot E)R$, where (E) denotes the H-ideal generated by E in R.*

(2) If B is an H-ideal of R and C is an H-ideal of B, then $(C)^3 \subseteq C$, where (C) denotes the H-ideal generated by C in R.

Proof. It is trivial. $\square$

Proposition 1.2 *(1) R is H-semiprime iff $(H \cdot a)R(H \cdot a) = 0$ always implies $a = 0$ for any $a \in R$.*

(2) R is H-prime iff $(H \cdot a)R(H \cdot b) = 0$ always implies $a = 0$ or $b = 0$ for any $a, b \in R$.

Proof. If R is an H-prime module algebra and $(H \cdot a)R(H \cdot b) = 0$ for $a, b \in R$, then $(a)^2(b)^2 = 0$, where (a) and (b) are the H-ideals generated by a and b in R respectively. Since R is H-prime, $(a) = 0$ or $(b) = 0$. Conversely, if B and C are H-ideals of R and $BC = 0$, then $(H \cdot a)R(H \cdot b) = 0$ and $a = 0$ or $b = 0$ for any $a \in B, b \in C$, which implies that $B = 0$ or $C = 0$, i.e. R is an H-prime module algebra.

Similarly, part (1) holds. $\square$

Proposition 1.3 *If $I \lhd_H R$ and I is an H-semiprime module algebra, then (1) $I \cap I^* = 0$; (2) $I_r = I_l = I^*$; (3) $I^* \lhd_H R$, where $I_r = \{a \in R \mid I(H \cdot a) = 0\}$, $I_l = \{a \in R \mid (H \cdot a)I = 0\}$, $I^* = \{a \in R \mid (H \cdot a)I = 0 = I(H \cdot a)\}$.*

Proof . For any $x \in I^* \cap I$, we have that $I(H \cdot x) = 0$ and $(H \cdot x)I(H \cdot x) = 0$. Since I is an H-semiprime module algebra, $x = 0$, i.e. $I \cap I^* = 0$.

To show $I^* = I_r$, we only need to show that $(H \cdot x)I = 0$ for any $x \in I_r$. For any $y \in I, h \in H$, let $z = (h \cdot x)y$. It is clear that $(H \cdot z)I(H \cdot z) = 0$. Since I is an H-semiprime module algebra, $z = 0$, i.e. $(H \cdot x)I = 0$. Thus $I^* = I_r$. Similarly, we can show that $I_l = I^*$.

Obviously, I^* is an ideal of R. For any $x \in I^*, h \in H$, we have $(H \cdot (h \cdot x))I = 0$. Thus $h \cdot x \in I^*$ by part (2), i.e. I^* is an H-ideal of R. $\square$

Definition 1.4 *$\mathcal{K}$ is called an H-(weakly)special class if*

(S1) $\mathcal{K}$ consists of H-(semiprime)prime module algebras.

(S2) For any $R \in \mathcal{K}$, if $0 \neq I \lhd_H R$ then $I \in \mathcal{K}$.

(S3) If R is an H-module algebra and $B \lhd_H R$ with $B \in \mathcal{K}$, then $R/B^* \in \mathcal{K}$, where $B^* = \{a \in R \mid (H \cdot a)B = 0 = B(H \cdot a)\}$.

It is clear that (S3) may be replaced by one of the following conditions:

(S3') If B is an essential H-ideal of R(i.e. $B \cap I \neq 0$ for any non-zero H-ideal I of R) and $B \in \mathcal{K}$, then $R \in \mathcal{K}$.

(S3") If there exists an H-ideal B of R with $B^* = 0$ and $B \in \mathcal{K}$, then $R \in \mathcal{K}$.

It is easy to check that if $\mathcal{K}$ is an H-special class, then $\mathcal{K}$ is an H-weakly special class.

Theorem 1.5 *If $\mathcal{K}$ is an H-weakly special class, then $r^{\mathcal{K}}(R) = \cap\{I \lhd_H R \mid R/I \in \mathcal{K}\}$, where $r^{\mathcal{K}}$ denotes the H-upper radical determined by $\mathcal{K}$.*

Proof. If I is a non-zero H-ideal of R and $I \in \mathcal{K}$, then $R/I^* \in \mathcal{K}$ by (S3) in Definition 1.4 and $I \not\subseteq I^*$ by Proposition 1.3. Consequently, it follows from [7, Proposition 5] that

$$r^{\mathcal{K}}(R) = \cap\{I \mid I \text{ is an } H\text{-ideal of } R \text{ and } R/I \in \mathcal{K}\} \quad . \quad \square$$

Definition 1.6 *If r is a hereditary H-radical(i.e. if R is an r-H-module algebra and B is an H-ideal of R, then so is B) and any nilpotent H-module algebra is an r-H-module algebra, then r is called a supernilpotent H-radical.*

Proposition 1.7 *r is a supernilpotent H-radical, then r is H-strongly hereditary, i.e. $r(I) = r(R) \cap I$ for any $I \lhd_H R$.*

Proof. It follows from [7, Proposition 4] . $\square$

Theorem 1.8 *If $\mathcal{K}$ is an H-weakly special class, then $r^{\mathcal{K}}$ is a supernilpotent H-radical.*

Proof. Let $r = r^{\mathcal{K}}$. Since every non-zero H-homomorphic image R' of a nilpotent H-module algebra R is nilpotent and is not H-semiprime, we have that R is an r-H-module algebra by Theorem 1.5. It remains to show that any H-ideal I of r-H-module algebra R is an r-H-ideal. If I is not an r-H-module algebra, then there exists an H-ideal J of I such that $0 \neq I/J \in \mathcal{K}$. By (S3), $(R/J)/(I/J)^* \in \mathcal{K}$. Let $Q = \{x \in R \mid (H \cdot x)I \subseteq J$ and $I(H \cdot x) \subseteq J \}$. It is clear that J and Q are H-ideals of R and $Q/J = (I/J)^*$. Since $R/Q \cong (R/J)/(Q/J) = (R/J)/(I/J)^*$ and R/Q is an r-H- module algebra, we have $(R/J)/(I/J)^*$ is an r-H-module algebra. Thus $R/Q = 0$ and $I^2 \subseteq J$, which contradicts that I/J is a non-zero H-semiprime module algebra. Thus I is an r-H-ideal. $\square$

Proposition 1.9 *R is H-semiprime iff for any $0 \neq a \in R$, there exists an H-m-sequence $\{a_n\}$ in R with $a_{7.1.1} = a$ such that $a_n \neq 0$ for all n.*

Proof. If R is H-semiprime, then for any $0 \neq a \in R$, there exist $b_1 \in R$, h_1 and $h_1' \in H$ such that $0 \neq a_2 = (h_1 \cdot a_1)b_1(h_1' \cdot a_1) \in (H \cdot a_1)R(H \cdot a_1)$ by Proposition 1.2, where $a_1 = a$. Similarly, for $0 \neq a_2 \in R$, there exist $b_2 \in R$ and h_2 and $h_2' \in H$ such that $0 \neq a_3 = (h_2 \cdot a_2)b_2(h_2' \cdot a_2) \in (H \cdot a_2)R(H \cdot a_2)$, which implies that there exists an H-m-sequence $\{a_n\}$ such that $a_n \neq 0$ for any natural number n. Conversely, it is trivial. $\square$

2 H-Baer radical

In this section, we give the characterization of H-Baer radical(H-prime radical) in many ways.

Theorem 2.1 *We define a property r_{Hb} for H-module algebras as follows: R is an r_{Hb}-H-module algebra iff every non-zero H-homomorphic image of R contains a non-zero nilpotent H-ideal; then r_{Hb} is an H-radical property.*

Proof. It is clear that every H-homomorphic image of r_{Hb}-H-module algebra is an r_{Hb}-H-module algebra. If every non-zero H-homomorphic image B of H-module algebra R contains a non-zero r_{Hb}-H-ideal I, then I contains a non-zero nilpotent H-ideal J. It is clear that (J) is a non-zero nilpotent H-ideal of B, where (J) denotes the H-ideal generated by J in B. Thus R is an r_{Hb}-H-module algebra. Consequently, r_{Hb} is an H-radical property. □

r_{Hb} is called H-prime radical or H-Baer radical.

Theorem 2.2 *Let*

$$\mathcal{E} = \{R \mid R \textit{ is a nilpotent } H\textit{-module algebra }\},$$

then $r_{\mathcal{E}} = r_{Hb}$, where $r_{\mathcal{E}}$ denotes the H-lower radical determined by $\mathcal{E}$.

Proof. If R is an r_{Hb}-H-module algebra, then every non-zero H-homomorphic image B of R contains a non-zero nilpotent H-ideal I. By the definition of the lower H-radical, I is an $r_{\mathcal{E}}$-H-module algebra. Consequently, R is an $r_{\mathcal{E}}$-H-module algebra. Conversely, since every nilpotent H-module algebra is an r_{Hb}-H-module algebra, $r_{\mathcal{E}} \leq r_{Hb}$. □

Proposition 2.3 *R is H-semiprime if and only if $r_{Hb}(R) = 0$.*

Proof. If R is H-semiprime with $r_{Hb}(R) \neq 0$, then there exists a non-zero nilpotent H-ideal I of $r_{Hb}(R)$. It is clear that H-ideal (I), which the H-ideal generated by I in R, is a non-zero nilpotent H-ideal of R. This contradicts that R is H-semiprime. Thus $r_{Hb}(R) = 0$. Conversely, if R is an H-module algebra with $r_{Hb}(R) = 0$ and there exists a non-zero nilpotent H-ideal I of R, then $I \subseteq r_{Hb}(R)$. We get a contradiction. Thus R is H-semiprime if $r_{Hb}(R) = 0$. □

Theorem 2.4 *If $\mathcal{K} = \{R \mid\ R$ is an H-prime module algebra$\}$, then $\mathcal{K}$ is an H-special class and $r_{Hb} = r^{\mathcal{K}}$.*

Proof. Obviously, $(S1)$ holds. If I is a non-zero H-ideal of an H-prime module algebra R and $BC = 0$ for H-ideals B and C of I, then $(B)^3(C)^3 = 0$ where (B) and (C) denote the H-ideals generated by B and C in R respectively. Since R is H-prime, $(B) = 0$ or $(C) = 0$, i.e. $B = 0$ or $C = 0$. Consequently, $(S2)$ holds. Now we shows that (S3) holds. Let B be an H-prime module algebra and be an H-ideal of R. If $JI \subseteq B^*$ for H-ideals I and J of R, then $(BJ)(IB) = 0$, where $B^* = \{x \in R \mid (H \cdot x)B = 0 = B(H \cdot x)\}$. Since B is an H-prime module algebra, $BJ = 0$ or $IB = 0$. Considering I and J are H-ideals, we have that $B(H \cdot J) = 0$ or $(H \cdot I)B = 0$. By Proposition 1.3, $J \subseteq B^*$ or $I \subseteq B^*$, which implies that R/B^* is an H-prime module algebra. Consequently, $(S3)$ holds and so $\mathcal{K}$ is an H-special class.

Next we show that $r_{Hb} = r^{\mathcal{K}}$. By Proposition 1.5, $r^{\mathcal{K}}(R) = \cap\{I \mid I$ is an H-ideal of R and $R/I \in \mathcal{K}\}$. If R is a nilpotent H-module algebra, then R is an $r^{\mathcal{K}}$-H-module algebra. It follows from Theorem 2.2 that $r_{Hb} \leq r^{\mathcal{K}}$. Conversely, if $r_{Hb}(R) = 0$, then R is an H-semiprime module algebra by Proposition 2.3. For any $0 \neq a \in R$, there exist $b_1 \in R$, $h_1, h_1' \in H$ such that $0 \neq a_2 = (h_1 \cdot a_1)b_1(h_1' \cdot a_1) \in (H \cdot a_1)R(H \cdot a_1)$, where $a_1 = a$. Similarly, for $0 \neq a_2 \in R$, there exist $b_2 \in R$ and $h_2, h_2' \in H$ such that $0 \neq a_3 = (h_2 \cdot a_2)b_2(h_2' \cdot a_2) \in (H \cdot a_2)R(H \cdot a_2)$. Thus there exists an H-m-sequence $\{a_n\}$ such that $a_n \neq 0$ for any natural number n. Let

$$\mathcal{F} = \{I \mid I \text{ is an } H\text{-ideal of } R \text{ and } I \cap \{a_1, a_2, \cdots\} = \emptyset\}.$$

By Zorn's Lemma, there exists a maximal element P in $\mathcal{F}$. If I and J are H-ideals of R and $I \not\subseteq P$ and $J \not\subseteq P$, then there exist natural numbers n and m such that $a_n \in I$ and $a_m \in J$. Since $0 \neq a_{n+m+1} = (h_{n+m} \cdot a_{n+m})b_{n+m}(h'_{n+m} \cdot a_{n+m}) \in IJ$, which implies that $IJ \not\subseteq P$ and so P is an H-prime ideal of R. Obviously, $a \notin P$, which implies that $a \notin r^{\mathcal{K}}(R)$ and $r^{\mathcal{K}}(R) = 0$. Consequently, $r^{\mathcal{K}} = r_{Hb}$. □

Theorem 2.5 $r_{Hb}(R) = W_H(R)$.

Proof. If $0 \neq a \notin W_H(R)$, then there exists an H-prime ideal P such that $a \notin P$ by the proof of Thoerem 2.4. Thus $a \notin r_{Hb}(R)$, which implies that $r_{Hb}(R) \subseteq W_H(R)$. Conversely, for any $x \in W_H(R)$, let $\bar{R} = R/r_{Hb}(R)$. Since $r_{Hb}(\bar{R}) = 0$, $\bar{R}$ is an H-semiprime module algebra by Proposition 2.3. By the proof of Theorem 2.4, $W_H(\bar{R}) = 0$. For an H-m-sequence $\{\bar{a}_n\}$ with $\bar{a}_1 = \bar{x}$ in $\bar{R}$, there exist $\overline{b}_n \in \overline{R}$ and $h_n, h_n' \in H$ such that

$$\overline{a}_{n+1} = (h_n \cdot \overline{a}_n)\overline{b}_n(h_n' \cdot \overline{a}_n)$$

for any natural number n. Thus there exists $a_n' \in R$ such that $a_1' = x$ and $a'_{n+1} = (h_n \cdot a_n')b_n(h_n' \cdot a_n')$ for any natural number n. Since $\{a_n'\}$ is an H-m-sequence with $a_1' = x$ in R, there exists a natural number k such that $a_k' = 0$. It is easy to show that $\overline{a}_n = \overline{a}_n'$ for any natural number n by induction. Thus $\bar{a}_k = 0$ and $\overline{x} \in W_H(\overline{R})$. Considering $W_H(\overline{R}) = 0$, we have $x \in r_{Hb}(R)$ and $W_H(R) \subseteq r_{Hb}(R)$. Therefore $W_H(R) = r_{Hb}(R)$. □

Definition 2.6 *We define an H-ideal N_α in H-module algebra R for every ordinal number α as follows:*

(i) $N_0 = 0$.

Let us assume that N_α is already defined for $\alpha \prec \beta$.

(ii) If $\beta = \alpha + 1, N_\beta/N_\alpha$ is the sum of all nilpotent H-ideals of R/N_α

(iii) If β is a limit ordinal number, $N_\beta = \sum_{\alpha \prec \beta} N_\alpha$.

By set theory, there exists an ordinal number τ such that $N_\tau = N_{\tau+1}$.

Theorem 2.7 $N_\tau = r_{Hb}(R) = \cap\{I \mid I$ *is an H-semiprime ideal of* $R\}$.

Proof. Let $D = \cap\{I \mid I$ is an H-semiprime ideal of $R\}$. Since R/N_τ has not any non-zero nilpotent H-ideal, we have that $r_{Hb}(R) \subseteq N_\tau$ by Proposition 2.3. Obviously, $D \subseteq r_{Hb}(R)$. Using transfinite induction, we can show that $N_\alpha \subseteq I$ for every H-semiprime ideal I of R and every ordinal number α (see the proof of [12, Theorem 3.7]). Thus $N_\tau \subseteq D$, which completes the proof. □

Definition 2.8 *Let $\emptyset \neq L \subseteq H$. An H-m-sequence $\{a_n\}$ in R is called an L-m-sequence with beginning a if $a_{7.1.1} = a$ and $a_{n+1} = (h_n.a_n)b_n(h'_n.a_n)$ such that $h_n, h'_n \in L$ for all n. For every L-m-sequence $\{a_n\}$ with $a_{7.1.1} = a$, there exists a natural number k such that $a_k = 0$, then R is called an L-m-nilpotent element, written as $W_L(R) = \{a \in R \mid a$ is an L-m-nilpotent element$\}$.*

Similarly, we have

Proposition 2.9 *If $L \subseteq H$ and $H = kL$, then*

(i) R is H-semiprime iff $(L.a)R(L.a) = 0$ always implies $a = 0$ for any $a \in R$.

(ii) R is H-prime iff $(L.a)R(L.b) = 0$ always implies $a = 0$ or $b = 0$ for any $a, b \in R$.

(iii) R is H-semiprime if and only if for any $0 \neq a \in R$, there exists an L-m-sequence $\{a_n\}$ with $a_1 = a$ such that $a_n \neq 0$ for all n.

(iv) $W_H(R) = W_L(R)$.

3 The H-module theoretical characterization of H-special radicals

If V is an algebra over k with unit and $x \otimes 1_V = 0$ always implies that $x = 0$ for any right k-module M and for any $x \in M$, then V is called a faithful algebra to tensor. For example, if k is a field, then V is faithful to tensor for any algebra V with unit.

In this section, we need to add the following condition: H is faithful to tensor.

We shall characterize H-Baer radical r_{Hb}, H-locally nil radical r_{Hl}, H-Jacobson radical r_{Hj} and H-Brown-McCoy radical r_{Hbm} by R-H-modules.

We can view every H-module algebra R as a sub-algebra of $R\#H$ since H is faithful to tensor. By computation, we have that

$$h \cdot a = \sum (1\#h_1)a(1\#S(h_2))$$

for any $h \in H, a \in R$, where S is the antipode of H.

Definition 3.1 *An R-H-module M is called an R-H-prime module if for M the following conditions are fulfilled:*

(i) $RM \neq 0$;

(ii) If x is an element of M and I is an H-ideal of R, then $I(Hx) = 0$ always implies $x = 0$ or $I \subseteq (0 : M)_R$, where $(0 : M)_R = \{a \in R \mid aM = 0\}$.

Definition 3.2 *We associate to every H-module algebra R a class $\mathcal{M}_R$ of R-H-modules. Then the class $\mathcal{M} = \cup \mathcal{M}_R$ is called an H-special class of modules if the following conditions are fulfilled:*

(M1) If $M \in \mathcal{M}_R$, then M is an R-H-prime module.

(M2) If I is an H-ideal of R and $M \in \mathcal{M}_I$, then $IM \in \mathcal{M}_R$.

(M3) If $M \in \mathcal{M}_R$ and I is an H-ideal of R with $IM \neq 0$, then $M \in \mathcal{M}_I$.

(M4) Let I be an H-ideal of R and $\bar{R} = R/I$. If $M \in \mathcal{M}_R$ and $I \subseteq (0 : M)_R$, then $M \in \mathcal{M}_{\bar{R}}$. Conversely, if $M \in \mathcal{M}_{\bar{R}}$, then $M \in \mathcal{M}_R$.

Let $\mathcal{M}(R)$ denote $\cap\{(0 : M)_R \mid M \in \mathcal{M}_R\}$, or R when $\mathcal{M}_R = \emptyset$.

Lemma 3.3 *(1) If M is an R-H-module, then M is an $R\#H$-module. In this case, $(0 : M)_{R\#H} \cap R = (0 : M)_R$ and $(0 : M)_R$ is an H-ideal of R;*

(2) R is a non-zero H-prime module algebra iff there exists a faithful R-H-prime module M;

(3) Let I be an H-ideal of R and $\bar{R} = R/I$. If M is an R-H-(resp. prime, irreducible)module and $I \subseteq (0 : M)_R$, then M is an $\overline{R}$-H-(resp. prime, irreducible)module (defined by $h \cdot (a + I) = h \cdot a$ and $(a + I)x = ax$). Conversely, if M is an $\bar{R}$-H-(resp. prime irreducible)module, then M is an R-H-(resp. prime, irreducible)module(defined by $h \cdot a = h \cdot (a + I)$ and $ax = (a + I)x$). In the both cases, it is always true that $R/(0 : M)_R \cong \overline{R}/(0 : M)_{\overline{R}}$;

(4) I is an H-prime ideal of R with $I \neq R$ iff there exists an R-H-prime module M such that $I = (0 : M)_R$;

(5) If I is an H-ideal of R and M is an I-H-prime module, then IM is an R-H-prime module with $(0 : M)_I = (0 : IM)_R \cap I$;

(6) If M is an R-H-prime module and I is an H-ideal of R with $IM \neq 0$, then M is an I-H-prime module;

(7) If R is an H-semiprime module algebra with one side unit, then R has a unit.

Proof. (1) Obviously, $(0 : M)_R = (0 : M)_{R\#H} \cap R$. For any $h \in H, a \in (0 : M)_R$, we see that $(h \cdot a)M = \sum(1\#h_1)a(1\#S(h_2))M \subseteq \sum(1\#h_1)aM = 0$ for any $h \in H, a \in R$. Thus $h \cdot a \in (0 : M)_R$, which implies $(0 : M)_R$ is an H-ideal of R.

(2) If R is an H-prime module algebra, view $M = R$ as an R-H-module. Obviously, M is faithful. If $I(H \cdot x) = 0$ for $0 \neq x \in M$ and an H-ideal I of R, then $I(x) = 0$ and $I = 0$, where (x) denotes the H-ideal generated by x in R. Consequently, M is a faithful R-H-prime module. Conversely, let M be a faithful R-H-prime module. If $IJ = 0$ for two H-ideals I and J of R with $J \neq 0$, then $JM \neq 0$ and there exists $0 \neq x \in JM$ such

that $I(Hx) = 0$. Since M is a faithful R-H-prime module, $I = 0$. Consequently, R is H-prime.

(3) If M is an R-H-module, then it is clear that M is a (left)$\overline{R}$-module and $h(\overline{a}x) = h(ax) = \sum(h_1 \cdot a)(h_2x) = \sum \overline{(h_1 \cdot a)}(h_2x) = \sum(h_1 \cdot \overline{a})(h_2x)$ for any $h \in H$, $a \in R$ and $x \in M$. Thus M is an $\overline{R}$-H-module. Conversely, if M is an $\overline{R}$-H-module, then M is an (left) R-module and

$$h(ax) = h(\overline{a}x) = \sum(h_1 \cdot \overline{a})(h_2x) = \sum \overline{h_1 \cdot a}(h_2x) = \sum(h_1 \cdot a)(h_2x)$$

for any $h \in H$, $a \in R$ and $x \in M$. This shows that M is an R-H-module.

Let M be an R-H-prime module and I be an H-ideal of R with $I \subseteq (0 : M)_R$. If $\overline{J}(Hx) = 0$ for $0 \neq x \in M$ and an H-ideal J of R, then $J(Hx) = 0$ and $J \subseteq (0 : M)_R$. This shows that $\bar{J} \subseteq (0 : M)_{\overline{R}}$. Thus M is an R-H-prime module. Similarly, we can show the other assert.

(4) If I is an H-prime ideal of R with $R \neq I$, then $\overline{R} = R/I$ is an H-prime module algebra. By Part (2), there exists a faithful $\overline{R}$-H-prime module M. By part (3), M is an R-H-prime module with $(0 : M)_R = I$. Conversely, if there exists a R-H-prime M with $I = (0 : M)_R$, then M is a faithful $\overline{R}$-H-prime module by part (3) and I is an H-prime ideal of R by part (2).

(5) First, we show that IM is an R-module. We define

$$a(\sum_i a_ix_i) = \sum_i (aa_i)x_i \tag{1}$$

for any $a \in R$ and $\sum_i a_ix_i \in IM$, where $a_i \in I$ and $x_i \in M$. If $\sum_i a_ix_i = \sum_i a_i'x_i'$ with a_i, $a_i' \in R$, x_i, $x_i' \in M$, let $y = \sum_i(aa_i)x_i - \sum_i(aa_i')x_i'$. For any $b \in I$ and $h \in H$, we see that

$$\begin{aligned} b(hy) &= \sum_i b\{h[(aa_i)x_i - (aa_i')x_i']\} \\ &= \sum_i \sum_{(h)} b\{[(h_1 \cdot (aa_i)](h_2x_i) - [h_1 \cdot (aa_i')](h_2x_i')\} \\ &= \sum_{(h)} \sum_i \{b[(h_1 \cdot a)(h_2 \cdot a_i)](h_3x_i)) - b[(h_1 \cdot a)(h_2 \cdot a_i')](h_3x_i')\} \\ &= \sum_{(h)} \sum_i b(h_1 \cdot a)[h_2(a_ix_i) - h_2(a_i'x_i')] \\ &= \sum_{(h)} b(h_1 \cdot a)h_2 \sum_i [a_ix_i - a_i'x_i'] = 0. \end{aligned}$$

Thus $I(Hy) = 0$. Since M is an I-H-prime module and $IM \neq 0$, we have that $y = 0$. Thus this definition in (1) is well-defined. It is easy to check that IM is an R-module. We see that

$$h(a \sum_i a_ix_i) \quad = \quad \sum_i h[(aa_i)x_i]$$

$$
\begin{aligned}
&= \sum_i \sum_h [h_1 \cdot (aa_i)][h_2 x_i] \\
&= \sum_i \sum_h [(h_1 \cdot a)(h_2 \cdot a_i)](h_3 x) \\
&= \sum_h (h_1 \cdot a) \sum_i (h_2 \cdot a_i)(h_3 x_i) \\
&= \sum_h (h_1 \cdot a)[h_2 \sum_i (a_i x_i)]
\end{aligned}
$$

for any $h \in H$ and $\sum_i a_i x_i \in IM$. Thus IM is an R-H-module.

Next, we show that $(0 : M)_I = (0 : IM)_R \cap I$. If $a \in (0 : M)_I$, then $aM = 0$ and $aIM = 0$, i.e. $a \in (0 : IM)_A \cap I$. Conversely, if $a \in (0 : IM)_R \cap I$, then $aIM = 0$. By part (1), $(0 : IM)_R$ is an H-ideal of R. Thus $(H \cdot a)IM = 0$ and $(H \cdot a)I \subseteq (0 : M)_I$. Since $(0 : M)_I$ is an H-prime ideal of I by part (4), $a \in (0 : M)_I$. Consequently, $(0 : M)_I = (0 : IM)_R \cap I$.

Finally, we show that IM is an R-H-prime module. If $RIM = 0$, then $RI \subseteq (0 : M)_R$ and $I \subseteq (0 : M)_R$, which contradicts that M is an I-H-prime module. Thus $RIM \neq 0$. If $J(Hx) = 0$ for $0 \neq x \in IM$ and an H-ideal J of R, then $JI(Hx) \subseteq J(Hx) = 0$. Since M is an I-H-prime module, $JI \subseteq (0 : M)_I$ and $J(IM) = 0$. Consequently, IM is an R-H-prime module.

(6) Obviously, M is an I-H-module. If $J(Hx) = 0$ for $0 \neq x \in M$ and an H-ideal J of I, then $(J)^3(Hx) = 0$ and $(J)^3 \subseteq (0 : M)_R$, where (J) denotes the H-ideal generated by J in R. Since $(0 : M)_R$ is an H-prime ideal of R, $(J) \subseteq (0 : M)_R$ and $J \subseteq (0 : M)_I$. Consequently, M is an I-H-prime module.

(7) We can assume that u is a right unit of R. We see that

$$(h \cdot (au - a))b = \sum (1\#h_1)(au - a)(1\#S(h_2))b = 0$$

for any $a, b \in R, h \in H$. Therefore $(H \cdot (au - a))R = 0$ and $au = a$, which implies that R has a unit. □

Theorem 3.4 *(1) If $\mathcal{M}$ is an H-special class of modules and $\mathcal{K} = \{ R \mid$ there exists a faithful R-H-module $M \in \mathcal{M}_R\}$, then $\mathcal{K}$ is an H-special class and $r^{\mathcal{K}}(R) = \mathcal{M}(R)$.*

(2) If $\mathcal{K}$ is an H-special class and $\mathcal{M}_R = \{ M \mid M$ is an R-H-prime module and $R/(0 : M)_R \in \mathcal{K}\}$, then $\mathcal{M} = \cup \mathcal{M}_R$ is an H-special class of modules and $r^{\mathcal{K}}(R) = \mathcal{M}(R)$.

Proof. (1) By Lemma 3.3(2), $(S1)$ is satisfied. If I is a non-zero H-ideal of R and $R \in \mathcal{K}$, then there exists a faithful R-H-prime module $M \in \mathcal{M}_R$. Since M is faithful, $IM \neq 0$ and $M \in \mathcal{M}_I$ with $(0 : M)_I = (0 : M)_R \cap I = 0$ by $(M3)$. Thus $I \in \mathcal{K}$ and $(S2)$ is satisfied. Now we show that $(S3)$ holds. If I is an H-ideal of R with $I \in \mathcal{K}$, then there exists a faithful I-H-prime module $M \in \mathcal{M}_I$. By $(M2)$ and Lemma 3.3(5), $IM \in \mathcal{M}_R$ and $0 = (0 : M)_I = (0 : IM)_R \cap I$. Thus $(0 : IM)_R \subseteq I^*$. Obviously, $I^* \subseteq (0 : IM)_R$. Thus

$I^* = (0 : IM)_R$. Using $(M4)$, we have that $IM \in \mathcal{M}_{\overline{R}}$ and IM is a faithful $\overline{R}$-H-module with $\overline{R} = R/I^*$. Thus $R/I^* \in \mathcal{K}$. Therefore $\mathcal{K}$ is an H-special class.

It is clear that

$$\{I \mid I \text{ is an } H\text{-ideal of } R \text{ and } R/I \in \mathcal{K}\} \quad = \quad \{(0 : M)_R \mid M \in \mathcal{M}_R\}.$$

Thus $r^{\mathcal{K}}(R) = \mathcal{M}(R)$.

(2) It is clear that $(M1)$ is satisfied. If I is an H-ideal of R with $M \in \mathcal{M}_I$, then M is an I-H-prime module with $I/(0 : M)_I \in \mathcal{K}$. By Lemma 3.3(5), IM is an R-H-prime module with $(0 : M)_I = (0 : IM)_R \cap I$. It is clear that

$$(0 : IM)_R = \{a \in R \mid (H \cdot a)I \subseteq (0 : M)_I \text{ and } I(H \cdot a) \subseteq (0 : M)_I\}$$

and

$$(0 : IM)_R/(0 : M)_I = (I/(0 : M)_I)^*.$$

Thus $R/(0 : IM)_R \cong (R/(0 : M)_I)/((0 : IM)_R/(0 : M)_I) = (R/(0 : M)_I)/(I/(0 : M)_I)^* \in \mathcal{K}$, which implies that $IM \in \mathcal{M}_R$ and $(M2)$ holds. Let $M \in \mathcal{M}_R$ and I be an H-ideal of R with $IM \neq 0$. By Lemma 3.3(6), M is an I-H-prime module and $I/(0 : M)_I = I/((0 : M)_R \cap I) \cong (I + (0 : M)_R)/(0 : M)_R$. Since $R/(0 : M)_R \in \mathcal{K}$, $I/(0 : M)_I \in \mathcal{K}$ and $M \in \mathcal{M}_I$. Thus $(M3)$ holds. It follows from Lemma 3.3(3) that $(M4)$ holds.

It is clear that

$$\{I \mid I \text{ is an } H\text{-ideal of } R \text{ and } 0 \neq R/I \in \mathcal{K}\} = \{(0 : M)_R \mid M \in \mathcal{M}_R\}.$$

Thus $r^{\mathcal{K}}(R) = \mathcal{M}(R)$. $\square$

Theorem 3.5 *Let $\mathcal{M}_R = \{ M \mid M$ is an R-H-prime module$\}$ for any H-module algebra R and $\mathcal{M} = \cup \mathcal{M}_R$. Then $\mathcal{M}$ is an H-special class of modules and $\mathcal{M}(R) = r_{Hb}(R)$.*

Proof. It follows from Lemma 3.3(3)(5)(6) that $\mathcal{M}$ is an H-special class of modules. By Lemma 3.3(2),

$$\{R \mid R \text{ is an } H\text{-prime module algebra with } R \neq 0\} =$$

$$\{R \mid \text{ there exists a faithful } R\text{-}H\text{-prime module } \}.$$

Thus $r_{Hb}(R) = \mathcal{M}(R)$ by Theorem 2.4(1). $\square$

Theorem 3.6 *Let $\mathcal{M}_R = \{ M \mid M$ is an R-H-irreducible module$\}$ for any H-module algebra R and $\mathcal{M} = \cup \mathcal{M}_R$. Then $\mathcal{M}$ is an H-special class of modules and $\mathcal{M}(R) = r_{Hj}(R)$, where r_{Hj} is the H-Jacobson radical of R defined in [7].*

Proof. If M is an R-H-irreducible module and $J(Hx) = 0$ for $0 \neq x \in M$ and an H-ideal J of R, let $N = \{m \in M \mid J(Hm) = 0\}$. Since $J(h(am)) = J(\sum_h (h_1 \cdot a)(h_2 m)) = 0$, $am \in N$ for any $m \in N, h \in H, a \in R$, we have that N is an R-submodule of M. Obviously, N is an H-submodule of M. Thus N is an R-H-submodule of M. Since $N \neq 0$, we have that $N = M$ and $JM = 0$, i.e. $J \subseteq (0 : M)_R$. Thus M is an R-H-prime module and (M1) is satisfied. If M is an I-H-irreducible module and I is an H-ideal, then IM is an R-H-module. If N is an R-H-submodule of IM, then N is also an I-H-submodule of M, which implies that $N = 0$ or $N = M$. Thus $(M2)$ is satisfied. If M is an R-H-irreducible module and I is an H-ideal of R with $IM \neq 0$, then $IM = M$. If N is an non-zero I-H-submodule of M, then IN is an R-H-submodule of M by Lemma 3.3(5) and $IN = 0$ or $IN = M$. If $IN = 0$, then $I \subseteq (0 : M)_R$ by the above proof and $IM = 0$. We get a contradiction. If $IN = M$, then $N = M$. Thus M is an I-H-irreducible module and $(M3)$ is satisfied.

It follows from Lemma 3.3(3) that (M4) holds. By Theorem 3.4(1), $\mathcal{M}(R) = r_{Hj}(R)$. $\square$

J.R. Fisher [7, Proposition 2] constructed an H-radical r_H by a common hereditary radical r for algebras, i.e. $r_H(R) = (r(R) : H) = \{a \in R \mid h \cdot a \in r(R) \text{ for any } h \in H\}$. Thus we can get H-radicals $r_{bH}, r_{lH}, r_{jH}, r_{bmH}$.

Definition 3.7 *An R-H-module M is called an R-H-BM-module, if for M the following conditions are fulfilled:*

(i) $RM \neq 0$;

(ii) If I is an H-ideal of R and $I \not\subseteq (0 : M)_R$, then there exists an element $u \in I$ such that $m = um$ for all $m \in M$.

Theorem 3.8 *Let $\mathcal{M}_R = \{ M \mid M$ is an R-H-BM-module$\}$ for every H-module algebra R and $\mathcal{M} = \cup \mathcal{M}_R$. Then $\mathcal{M}$ is an H-special class of modules.*

Proof. It is clear that M satisfies (M_1) and (M_4). To prove (M_2) we exhibit: if $I \lhd_H R$ and $M \in \mathcal{M}_I$, then M is an I-H-prime module and IM is an R-H-prime module. If J is an H-ideal of R with $J \not\subseteq (0 : M)_R$, then JI is an H-ideal of I with $JI \not\subseteq (0 : M)_I$. Thus there exists an element $u \in JI \subseteq J$ such that $um = m$ for every $m \in M$. Hence $IM \in \mathcal{M}_R$.

To prove (M_3), we exhibit: if $M \in \mathcal{M}_R$ and I is an H-ideal of R with $IM \neq 0$. If J is an H-ideal of I with $J \not\subseteq (0 : M)_I$, then $(J) \not\subseteq (0 : M)_R$, where (J) is the H-ideal generated by J in R. Thus there exists an elements $u \in (J)$ such that $um = m$ for every $m \in M$. Moreover,

$$m = um = uum = uuum = u^3 m$$

and $u^3 \in J$. Thus $M \in \mathcal{M}_I$. $\square$

Proposition 3.9 *If M is an R-H-BM-module, then $R/(0:M)_R$ is an H-simple module algebra with unit.*

Proof. Let I be any H- ideal of R with $I \not\subseteq (0:M)_R$. Since M is an R-H-BM-module, there exists an element $u \in I$ such that $uam = am$ for every $m \in M, a \in R$. It follows that $a - ua \in (0:M)_R$, whence $R = I + (0:M)_R$. Thus $(0:M)_R$ is a maximal H-ideal of R. Therefore $R/(0:M)_R$ is an H-simple module algebra.

Next we shall show that $R/(0:M)_R$ has a unit. Now $R \not\subseteq (0:M)_R$, since $RM \neq 0$. By the above proof, there exists an element $u \in R$ such that $a - ua \in (0:M)_R$ for any $a \in R$. Hence $R/(0:M)_R$ has a left unit. Furthermore, by Lemma 3.7 (7) it has a unity element. □

Proposition 3.10 *If R is an H-simple-module algebra with unit, then there exists a faithful R-H-BM-module.*

Proof. Let $M = R$. It is clear that M is a faithful R-H-BM- module. □

Theorem 3.11 *Let $\mathcal{M}_R = \{\ M \mid M$ is an R-H-BM-module$\}$ for every H-module algebra R and $\mathcal{M} = \cup \mathcal{M}_R$. Then $r_{Hbm}(R) = \mathcal{M}(R)$, where r_{Hbm} denotes the H-upper radical determined by $\{R \mid R$ is an H-simple module algebra with unit $\}$.*

Proof. By Theorem 3.8, $\mathcal{M}$ is an H-special class of modules. Let

$$\mathcal{K} = \{R \mid \text{ there exists a faithful } R\text{-}H\text{-}BM\text{-module } \}.$$

By Theorem 3.4(1), $\mathcal{K}$ is an H-special class and $r^{\mathcal{K}}(R) = \mathcal{M}(R)$. Using Proposition 3.9 and 3.10, we have that

$$\mathcal{K} = \{R \mid R \text{ is an } H\text{-simple module algebra with unit } \}.$$

Therefore $\mathcal{M}(R) = r_{Hbm}(R)$. □

Assume that H is a finite-dimensional semisimple Hopf algebra with $t \in \int_H^l$ and $\epsilon(t) = 1$. Let

$$G_t(a) = \{z \mid z = x + (t.a)x + \sum(x_i(t.a)y_i + x_iy_i) \ \text{ for all } \ x_i, y_i, x \in R\}.$$

R is called an r_{gt}-H-module algebra, if $a \in G_t(a)$ for all $a \in R$.

Theorem 3.12 *r_{gr} is an H-radical property of H-module algebra and $r_{gt} = r_{Hbm}$.*

Proof. It is clear that any H-homomorphic image of r_{gt}-H- module algebra is an r_{gt}-H-module algebra. Let

$$N = \sum\{I \lhd_H \mid I \text{ is an } r_{gt}\text{-}H\text{-ideal of } R\}.$$

Now we show that N is an r_{gt}-H-ideal of R. In fact, we only need to show that $I_1 + I_2$ is an r_{gt}-H-ideal for any two r_{gt}-H-ideals I_1 and I_2. For any $a \in I_1, b \in I_2$, there exist $x, x_i, y_i \in R$ such that

$$a = x + (t \cdot a)x + \sum_i (x_i(t \cdot a)y_i + x_i y_i).$$

Let

$$c = x + (t \cdot (a+b))x + \sum x_i(t \cdot (a+b))y_i + x_i y_i \qquad \in G_t(a+b).$$

Obviously,

$$a + b - c = b - (t \cdot b)x - \sum x_i(t \cdot b)y_i \qquad \in I_2.$$

Thus there exist $w, u_j, v_j \in R$ such that

$$a + b - c = w + (t \cdot (a+b-c))w + \sum_j (u_j(t \cdot (a+b-c))v_j + u_j v_j).$$

Let $d = (t \cdot (a+b))w + w + \sum_j (u_j(t \cdot (a+b))v_j + u_j v_j)$ and $e = c - \sum_j u_j(t \cdot c)v_j - (t \cdot c)w$. By computation, we have that

$$a + b = d + e.$$

Since $c \in G_t(a+b)$ and $d \in G_t(a+b)$, we get that $e \in G_(a+b)$ and $a + b \in G_t(a+b)$, which implies that $I_1 + I_2$ is an r_{gt}-H-ideal.

Let $\bar{R} = R/N$ and $\bar{B}$ be an r_{gt}-H-ideal of $\bar{R}$. For any $a \in B$, there exist $x, x_i, y_i \in R$ such that

$$\bar{a} = \bar{x} + (t \cdot \bar{a})\bar{x} + \sum (\bar{x}_i(t \cdot \bar{a})\bar{y}_i + \bar{x}_i \bar{y}_i)$$

and

$$x + (t \cdot a)x + \sum (x_i(t \cdot a)y_i + x_i y_i) - a \in N.$$

Let

$$c = x + (t \cdot a)x + \sum (x_i(t \cdot a)y_i + x_i y_i) \in G_t(a).$$

Thus there exist $w, u_j, v_j \in R$ such that

$$a - c = (t \cdot (a-c))w + w + \sum (u_j(t \cdot (a-c))v_j + u_j v_j)$$

and

$$a = (t \cdot a)w + w + \sum u_j(t \cdot a)v_j + u_j v_j + c - (t \cdot c)w - \sum u_j(t \cdot c)v_j \qquad \in G_t(a),$$

which implies that B is an r_{gt}-H-ideal and $\bar{B} = 0$. Therefore r_{gt} is an H -radical property. □

Proposition 3.13 *If R is an H-simple module algebra, then $r_{gr}(R) = 0$ iff R has a unit.*

Proof. If R is an H-simple module algebra with unit 1, then $-1 \notin G_t(-1)$ since

$$x + (t \cdot (-1))x + \sum (x_i(t \cdot (-1))y_i + x_i y_i) = 0$$

for any $x, x_i, y_i \in R$. Thus R is r_{gt}-H-semisimple. Conversely, if $r_{gt}(R) = 0$, then there exists $0 \neq a \notin G_t(a)$ and $G_t(a) = 0$, which implies that $ax + x = 0$ for any $x \in R$. It follows from Lemma 9.3.3 (7) that R has a unit. □

Theorem 3.14 $r_{gt} = r_{Hbm}$.

Proof. By Proposition 3.13, $r_{gt}(R) \subseteq r_{Hbm}(R)$ for any H-module algebra R. It remains to show that if $a \notin r_{gt}(R)$ then $a \notin r_{Hbm}(R)$. Obviously, there exists $b \in (a)$ such that $b \notin G_t(b)$, where (a) denotes the H-ideal generated by a in R. Let

$$\mathcal{E} = \{I \lhd_H R \mid G_t(b) \subseteq I, b \notin I\}.$$

By Zorn's Lemma, there exists a maximal element P in $\mathcal{E}$. P is a maximal H-ideal of R, for, if Q is an H-ideal of R with $P \subseteq Q$ and $P \neq Q$, then $b \in Q$ and $x = -bx + (bx + x) \in Q$ for any $x \in R$. Consequently, R/P is an H-simple module algebra with $r_{gt}(R/P) = 0$. It follows from Proposition 3.13 that R/P is an H-simple module algebra with unit and $r_{Hbm}(R) \subseteq P$. Therefore $b \notin r_{Hbm}(R)$ and so $a \notin r_{Hbm}(R)$. □

Definition 3.15 *Let I be an H-ideal of H-module algebra R, N be an R-H-submodule of R-H-module M. (N, I) are said to have "L-condition", if for any finite subset $F \subseteq I$, there exists a positive integer k such that $F^k N = 0$.*

Definition 3.16 *An R-H-module M is called an R-H-L-module, if for M the following conditions are fulfilled:*

(i) $RM \neq 0$.

(ii) For every non-zero R-H-submodule N of M and every H-ideal I of R, if (N, I) has "L-condition", then $I \subseteq (0 : M)_R$.

Proposition 3.17 *If M is an R-H-L-module, then $R/(0 : M)_R$ is an r_{lH}-H-semisimple and H-prime module algebra.*

Proof. If M is an R-H-L-module, let $\bar{R} = R/(0 : M)_R$. Obviously, $\bar{R}$ is H-prime. If $\bar{B}$ is an r_{lH}-H-ideal of $\bar{R}$, then (M, B) has "L-condition" in R-H-module M, since for any finite subset F of B, there exists a natural number n such that $F^n \subseteq (0 : M)_R$ and $F^n M = 0$. Consequently, $B \subseteq (0 : M)_R$ and $\bar{R}$ is r_{lH}-semisimple. □

Proposition 3.18 *R is a non-zero r_{lH}-H-semisimple and H-prime module algebra iff there exists a faithful R-H-L-module.*

Proof. If R is a non-zero r_{lH}-H-semisimple and H-prime module algebra, let $M = R$. Since R is an H-prime module algebra, $(0 : M)_R = 0$. If (N, B) has "L-condition" for non-zero R-H-submodule of M and H-ideal B, then, for any finite subset F of B, there exists an natural number n, such that $F^n N = 0$ and $F^n(NR) = 0$, which implies that $F^n = 0$ and B is an r_{lH}-H-ideal, i.e. $B = 0 \subseteq (0 : M)_R$. Consequently, M is a faithful R-H-L- module.

Conversely, if M is a faithful R-H-L-module, then R is an H-prime module algebra. If I is an r_{lH}-H-ideal of R, then (M, I) has "L-condition", which implies $I = 0$ and R is an r_{lH}-H-semisimple module algebra. □

Theorem 3.19 *Let $\mathcal{M}_R = \{\ M \mid M$ is an R-H-L-module$\}$ for any H-module algebra R and $\mathcal{M} = \cup \mathcal{M}_R$. Then $\mathcal{M}$ is an H-special class of modules and $\mathcal{M}(R) = r_{Hl}(R)$, where $\mathcal{K} = \{R \mid R$ is an H-prime module algebra with $r_{lH}(R) = 0\}$ and $r_{Hl} = r^{\mathcal{K}}$.*

Proof. Obviously, $(M1)$ holds. To show that $(M2)$ holds, we only need to show that if I is an H-ideal of R and $M \in \mathcal{M}_I$, then $IM \in \mathcal{M}_R$. By Lemma 3.3(5), IM is an R-H-prime module. If (N, B) has the "L-condition" for non-zero R-H-submodule N of IM and H-ideal B of R, i.e. for any finite subset F of B, there exists a natural number n such that $F^n N = 0$, then (N, BI) has "L-condition" in I-H-module M. Thus $BI \subseteq (0 : M)_I = (0 : IM)_R \cap I$. Considering $(0 : IM)_R$ is an H-prime ideal of R, we have that $B \subseteq (0 : IM)_R$ or $I \subseteq (0 : IM)_R$. If $I \subseteq (0 : IM)_R$, then $I^2 \subseteq (0 : M)_I$ and $I \subseteq (0 : M)_I$, which contradicts $IM \neq 0$. Therefore $B \subseteq (0 : IM)_R$ and so IM is an R-H-L- module.

To show that $(M3)$ holds, we only need to show that if $M \in \mathcal{M}_R$ and $I \lhd_H R$ with $IM \neq 0$, then $M \in \mathcal{M}_I$. By Lemma 3.3(6), M is an I-H-prime module. If (N, B) has the "L-condition" for non-zero I-H-submodule N of M and H-ideal B of I, then IN is an R-H-prime module and $(IN, (B))$ has "L-condition" in R-H-module M, since for any finite subset F of (B), $F^3 \subseteq B$ and there exists a natural number n such that $F^{3n} IN \subseteq F^{3n} N = 0$, where (B) is the H-ideal generated by B in R. Therefore, $(B) \subseteq (0 : M)_R$ and $B \subseteq (0 : M)_I$, which implies $M \in \mathcal{M}_I$.

Finally, we show that $(M4)$ holds. Let $I \lhd_H R$ and $\bar{R} = R/I$. If $M \in \mathcal{M}_R$ and $I \subseteq (0 : M)_R$, then M is an $\bar{R}$-H- prime module. If $(N, \bar{B})$ has "L-condition" for H-ideal $\bar{B}$ of $\bar{R}$ and $\bar{R}$-H-submodule N of M, then subset $F \subseteq B$ and there exists a natural number n such that $F^n N = (\bar{F})^n N = 0$. Consequently, $M \in \mathcal{M}_{\bar{R}}$. Conversely, if $M \in \mathcal{M}_{\bar{R}}$, we can similarly show that $M \in \mathcal{M}_R$.

The second claim follows from Proposition 3.18 and Theorem 3.4(1). □

Theorem 3.20 $r_{Hl} = r_{lH}$.

Proof. Obviously, $r_{lH} \leq r_{Hl}$. It remains to show that $r_{Hl}(R) \neq R$ if $r_{lH}(R) \neq R$. There exists a finite subset F of R such that $F^n \neq 0$ for any natural number n. Let

$$\mathcal{F} = \{I \mid I \text{ is an } H\text{-ideal of } R \text{ with } F^n \not\subseteq I \text{ for any natural number } n\}.$$

By Zorn's lemma, there exists a maximal element P in $\mathcal{F}$. It is clear that P is an H-prime ideal of R. Now we show that $r_{lH}(R/P) = 0$. If $0 \neq B/P$ is an H-ideal of R/P, then there exists a natural number m such that $F^m \subseteq B$. Since $(F^m + P)^n \neq 0 + P$ for any natural number n, we have that B/P is not locally nilpotent and $r_{lH}(R/P) = 0$. Consequently, $r_{Hl}(R) \neq R$. □

In fact, all of the results hold in braided tensor categories determined by (co)quasitriangular structure.

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